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Least Squares Nonlinear Parameter Estimation by the Iterative Continuation Method

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Introduction

THE principle of least squares is widely used for estimating the unknown parameters of a nonlinear analytic model from a discrete sequence of noisy observations. In applying this principle, estimates of the parameters are determined by minimizing an error function based on the squares of the deviations between the computed response and the measured response. Minimization of the error function may be performed by several iterative procedures such as the linearization method.[1,2] This method has the advantage of a fast convergence, however, it necessitates good starting values for converging. It is proposed here to use a continuation method for finding good first approximations for the linearization procedure. Continuation methods are of two kinds: continuous or iterative.[3] The proposed method is of the second type. Its convergence properties will be illustrated by a numerical example.

A continuation method has been recently used for fitting a differential equation to experimental data.[4] The unknown parameters were determined by solving a nonlinear multipoint boundary-value problem. The solution is based on a continuation method of the continuous type.

Analysis

The system to identify is described by the equation

$$y = f(x, \gamma) \tag{1}$$

where f is a nonlinear function of the unknown parameter vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ and of the independent variable x.

Noisy measurements b_m are made on the variable $y(x)$ at stations x_m, $m = 1, 2, \ldots, N$. The estimation problem is to find the parameter vector $\hat{\gamma}$ which minimizes the function

$$J(\gamma) = \sum_{m=1}^{N} [b_m - f(x_m, \gamma)]^2 \tag{2}$$

Received April 2, 1974; revision received June 25, 1974. The authors are grateful to E. Wasserstrom and J. Rom for helpful discussions.

Index category: Computer Technology and Computer Simulation Techniques.

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The concept of continuation will be applied to solve this minimum problem. As said in Ref. 5, "continuation refers to the situation in which a problem depends on a parameter and in which the solution to the problem for one value of the parameter is used to find the solution for a nearby value of the parameter." Sometimes, this parameter t appears in the definition of the problem and the application of the continuation method is straightforward. However, if the parameter t does not appear explicitly in the problem, it is necessary in applying the continuation method to introduce it.

An auxiliary parameter t will be introduced in the problem in the following manner. Let γ^0 be an initial approximation of the minimizing vector $\hat{\gamma}$ of the function $J(\gamma)$. Define the function

$$S(\gamma, t) = tJ(\gamma) + (1-t)\sum_{m=1}^{N} [c_m(t) - f(x_m, \gamma)]^2 \tag{3}$$

where $J(\gamma)$ is defined in Eq. (2) and

$$c_m(t) = tb_m + (1-t)f(x_m, \gamma^0) \tag{4}$$

The functions $c_m(t)$ appear in Eq. (11) of Wasserstrom's paper.[4] They are similarly used in the present formulation.

The problem $P(t)$ is now considered. The problem is to find the minimizing vector $\hat{\gamma}(t)$ of the function $S(\gamma, t)$ for t in the interval $[0, 1]$. At $t = 0$, this problem is easily solved. The function $S(\gamma, 0)$ is

$$S(\gamma, 0) = \sum_{m=1}^{N} [f(x_m, \gamma^0) - f(x_m, \gamma)]^2 \tag{5}$$

The minimizing vector of $S(\gamma, 0)$ is trivial,

$$\hat{\gamma}(0) = \gamma^0 \tag{6}$$

At $t = 1$, the following condition applies:

$$S(\gamma, 1) = J(\gamma) \tag{7}$$

Therefore, $P(1)$ coincides with the original minimum problem and

$$\hat{\gamma}(1) = \hat{\gamma} \tag{8}$$

The desired solution $\hat{\gamma}(1)$ is next computed by a continuation procedure which starting from γ^0 at $t = 0$ "continues" this initial solution and yields $\hat{\gamma}$ at $t = 1$. Using the discrete version of the continuation method, the interval $0 \leq t \leq 1$ is subdivided into M equal parts so that $t_i = i/M$, $i = 1, 2, \ldots, M$. Each problem $P(t_i)$ will be solved by the linearization method. In applying this latter method, the solution $\hat{\gamma}(t_{i-1})$ to the problem $P(t_{i-1})$ is chosen as a starting value for solving the next problem $P(t_i)$. In particular, $\hat{\gamma}(0) = \gamma^0$ is taken as a first approximation for solving the problem $P(t_1)$.

The solution of the problem $P(t_i)$ by the linearization method is next described. For reducing the notation define

$$\hat{\gamma}(t_i) = \gamma^i \qquad i = 1, 2, \ldots, M$$

$$f_l(x, \gamma) = \partial f(x, \gamma)/\partial \gamma_l \qquad l = 1, 2, \ldots, k$$

Assume that γ^{i-1} is close to γ^i, then

$$\gamma^i \sim \gamma^{i-1} + \Delta\gamma \tag{9}$$

where $\Delta\gamma = (\Delta\gamma_1, \Delta\gamma_2, \ldots, \Delta\gamma_k)$ is a vector of small corrections. Linearize the function $f(x, \gamma)$ about γ^{i-1}

$$f(x, \gamma) \sim f(x, \gamma^{i-1}) + \sum_{l=1}^{k} f_l(x, \gamma^{i-1})\Delta\gamma_l \tag{10}$$

Substituting Eq. (10) into Eq. (3) yields the function

$$SS(\Delta\gamma, t_i) = t_i \sum_{m=1}^{N} [b_m - f(x_m, \gamma^{i-1}) - \sum_{l=1}^{k} f_l(x_m, \gamma^{i-1})\Delta\gamma_l]^2 + (1-t_i)\sum_{m=1}^{N} [c_m(t_i) - f(x_m, \gamma^{i-1}) - \sum_{l=1}^{k} f_l(x_m, \gamma^{i-1})\Delta\gamma_l]^2 \tag{11}$$

$SS(\Delta\gamma, t_i)$ is a quadratic form in $\Delta\gamma$ whose minimizer is

$$\Delta\gamma = A^{-1}B \tag{12}$$

The elements of $A(k \times k)$ and $B(k \times 1)$ are given by

$$A_{jh} = \sum_{m=1}^{N} f_h(x_m, \gamma^{i-1}) f_j(x_m, \gamma^{i-1}) \tag{13}$$

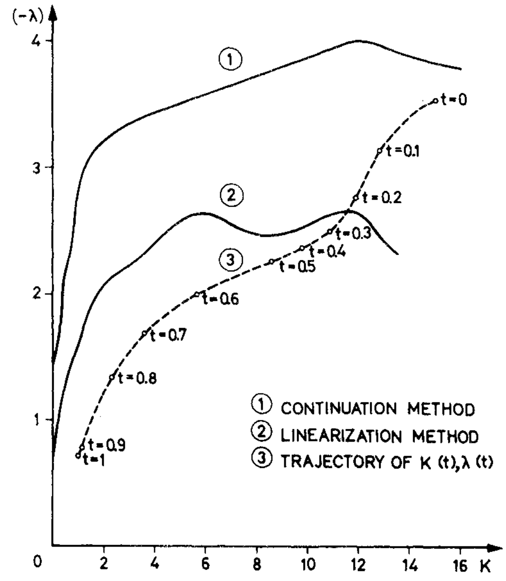

Fig. 1 Comparison of the domains of convergence of the linearization and the continuation methods.

$$B_j = \sum_{m=1}^{N} t_i(2-t_i)b_m + (1-t_i)^2 f(x_m, \gamma^0) - f(x_m, \gamma^{i-1})] \; f_j(x_m, \gamma^{i-1}) \qquad (14)$$

As previously stated, γ^{i-1} is taken as the first approximation of γ^i. Hopefully, a better approximation of γ^i is obtained by adding $\Delta\gamma$ (Eq. 12) to γ^{i-1} and the iteration process is continued until it converges to $\gamma^i \cdot \gamma^i$ is next taken as an initial approximation for solving the next problem $P(t_{i+1})$ by the precedent linearization procedure. This process is continued until $t = 1$ where one finds the desired solution $\hat{\gamma}$.

Numerical Example

The convergence properties of the proposed method and of the linearization method were compared in analysing the linear pure pitching motion of a missile. The equation of this motion is of the form:

$$y = Ke^{\lambda x} \cos(\omega x + \phi) \qquad (15)$$

where y denotes the angle of attack, and x the distance (speed × time). The damping λ and the pulsation ω are related to aerodynamic coefficients of the missile while the amplitude K and the phase ϕ depend on initial conditions. The problem is to determine the parameter vector:

$$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \qquad (16)$$

where $\gamma_1 = K$, $\gamma_2 = \lambda$, $\gamma_3 = \omega$, $\gamma_4 = \phi$ from a set of measurements. For testing the present approach, data were generated by a computer simulation of the missile equation (15). Gaussian noise with a standard deviation $\sigma = 0.02$ was added to the data. The parameters used in the simulation were as follows:

$$K_e = 1.00 \qquad \lambda_e = -0.70$$
$$\omega_e = 2.00 \qquad \phi_e = -0.0 \qquad (17)$$

Thirty equally spaced data points were generated. The distance between two successive points was $\Delta x = 0.2$.

These simulated data were analysed by applying the continuation procedure and the linearization procedure. The same starting approximations γ^0 were used for both methods and convergence within a margin of error allowing for the presence of noise to the solution given by Eq. (17) was tested.

The domains of convergence of the continuation method and of the linearization method were compared in the plane (K, λ). These domains were defined by choosing the initial guess of the parameter vector of the form:

$$\gamma^0 = (K^0, \lambda^0, \omega_e, \phi_e) \qquad (18)$$

Figure 1 presents the results of the computations. For physical reasons, only positive starting values of K and $(-\lambda)$ were considered. Figure 1 shows clearly that the domain of convergence of the linearization method is enlarged by the continuation algorithm.

Let $\hat{\gamma}(t) = [\hat{K}(t), \hat{\lambda}(t), \hat{\omega}(t), \hat{\phi}(t)]$ be the solution of the minimum problem $P(t)$. Also shown in Fig. 1 is the trajectory of the point $[\hat{K}(t), \hat{\lambda}(t)]$ with the initial values at $t = 0$

$$\hat{K}(0) = 14.99 \qquad \hat{\lambda}(0) = -3.53 \qquad (19)$$

well outside of the domain of convergence of the method of linearization. The following values were obtained at $t = 1$:

$$\hat{K} = 0.98 \quad \hat{\lambda} = 0.69 \quad \hat{\omega} = 2.02 \quad \hat{\phi} = 0.04 \qquad (20)$$

The computation time for that case was 7 sec on the IBM 370/165.

The deviations of the computed parameters (20) from the true values (17) are due to the presence of noise. The exact solution was indeed recovered when the noise was suppressed from the simulated data. It may be noted that the accuracies of the estimates by the linearization method and the continuation method are equal as both methods find the minimum $\hat{\gamma}$ of the same error function $J(\gamma)$ defined in Eq. (4).

The preceding results may seem particular as the sequence of the iterations started with two parameters initially equal to their exact values. Other experiments have been made in order to show that the observed increase of the domain of convergence of the linearization method due to the application of the continuation method is not accidental. For example, the convergence properties of the two methods have been compared by setting initial starting values on the straight line

$$K^0 = K_e(1+u) \qquad \lambda^0 = \lambda_e(1+1.5u)$$
$$\omega^0 = 1.50 \qquad \phi^0 = 0.50$$

where u is a positive parameter. The limit points of convergence were, respectively, for the linearization method $K^0 = 2.43$, $\lambda^0 = -2.132$, $\omega^0 = 1.5$, $\phi^0 = 0.5$ and for the continuation method $K^0 = 3.27$, $\lambda^0 = -2.97$, $\omega^0 = 1.5$, $\phi^0 = 0.5$. The superior convergence properties of the continuation algorithm also appear clearly in that example.

Conclusions

An iterative method of continuation has been formulated for finding the least-squares estimates of unknown parameters of nonlinear analytically modeled systems. The continuation algorithm which has been presented uses at intermediate steps the linearization method. This combination yields an increase in the convergence region of the linearization method. It provides a means of finding a good starting solution for the linearization method. The drawback of the continuation method is that it is rather slow since many intermediate problems have to be solved in order to obtain the solution. Further work is needed to improve the present algorithm for reducing the computer time and to extend it for identifying systems described by differential equations from noisy observations.

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